Efficient Computation of Mode-Shape Derivatives for Large Dynamic Systems

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This paper is concerned with the simplified numerical computation of eigenvector derivatives for very large problems with symmetric, banded matrices, such as those ordinarily encountered in structural dynamic systems. Starting with a brief review of existing alternate computation schemes, the paper focuses on Nelson's work, which is at present generally accepted as the most computationally efficient method. Two competitive and more traditional alternatives to Nelson's approach, one direct and one iterative, are offered. No claim is made that either of these methods are superior to Nelson's, but they do offer viable and more conventional alternatives. Also discussed is the common case of repeated, or closely spaced, eigenvalues, which was not treated by Nelson. A procedure for extending Nelson's ideas for such cases is proposed.

Nomenclature

$[A],[B]$	= large, banded, $n\times n$ symmetric matrices
$[\backslash A_{11}\backslash]$	= see item 3 following Eq. (6)
$\{A_{12}\}$	= see item 5 following Eq. (6)
$\lfloor A_{21}\rfloor$	= see item 4 following Eq. (6)
A_{22}	= see item 6 following Eq. (6)
b	= see Eq. (4)
c_1,c_2,c_3,c_i	= see Eqs. (18), (42), and (47)
$[D]$	= repeated eigenvalue derivative matrix, Eq. (35)
d_1,d_2,d_3	= see Eqs. (42) and (47)
e_1,e_2,e_3	= see Eq. (47)
$\{\bar{F}\}$	= see item 2 following Eq. (6)
$\{F_i\},\{G_i\}$	= see Eqs. (3) and (31)
i,j	= subscript indices
L	= multiplicity of eigenvalue
$[L]$	= lower triangular matrix, Eq. (10)
m	= bandwidth of $([A]-\lambda_i[B])$
n	= number of rows or columns in $[A]$ and $[B]$
r	= independent parameter upon which one or more elements of $[A]$ and $[B]$ depend
$\bar{r}$	= iteration index (number of iterations required for convergence)
t	= computer computation time
$\{u\},\{v\}$	= see Eqs. (11), (12), and (13)
$\{V_i\}$	= see Eqs. (18) and (47)
$[W]$	= modal matrix for repeated eigenvalue
$\{x_i\}$	= eigenvector whose derivative is sought
$\{\bar{x}_i'\}$	= see item 1 following Eq. (6)
x_p'	= see item 1 following Eq. (6)
$\{y_i\},\{z_i\}$	= see Eqs. (24) and (28)
α	= see Eq. (28)
ϵ	= convergence tolerance
$\{\zeta\}$	= see Eq. (34)
$\lambda_{G\max}$	= maximum eigenvalue of $[G]$, see Eq. (46)
λ_i	= eigenvalue, Eq. (1)

Superscripts

$(\ )'$	= derivative with respect to r
T	= transpose of matrix or vector

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Introduction

EIGENPARAMETER derivatives are extremely useful for determining the sensitivity of dynamic responses to system parameter variations. Such information is used regularly in structural design and optimization and to improve the correlation between analyses and experimental results.

While the direct computation of *eigenvalue* derivatives for large linear algebraic systems has been understood for quite some time,[1] the efficient calculation of *eigenvector* derivatives is still being addressed in the current literature.[2] The main reason for this is that the algebraic equations obtained upon differentiating the eigenproblem relationships, including the eigenvector normalization condition, result in an overdetermined, yet consistent, banded system of equations. The problem then becomes one of solving this system efficiently when large numbers of equations are involved. The problem is further compounded when repeated eigenvalues occur (e.g., as with an axisymmetric structure).

Some analysts who have approached this problem[1-4] resorted to eigenvector expansion methods, using a reduced number of modes when the original system was large. This could involve obtaining a sizable number of eigenvectors when only a single eigenvector's derivatives was required and it also left open the question of solution accuracy. An alternate approach automatically removes the redundancy in the overdetermined gradient equation system.[1] However, it requires premultiplication by the transpose of the coefficient matrix and involves significant additional computations that tend to destroy the bandedness of the original system.

Fortunately, these works were significantly improved upon a decade ago by Nelson.[5] He showed how to efficiently remove the singularity in the governing algebraic equations when the stiffness matrix was nonsingular and no repeated eigenvalues existed. A significant achievement of his work was that it did not destroy the bandedness of the original system matrices.

The present paper builds upon the above-cited ideas in Nelson's seminal work[5] and offers two alternatives to his clever idea of decomposing the eigenvector gradient $\{x_i'\}$ into a combination of a vector $\{V_i\}$ that satisfies an underdetermined system and the eigenvector $\{x_i\}$ whose gradient is sought. The present work also considers the problem of repeated eigenvalues and shows how to treat them in the spirit of Nelson's nonrepeated root method.

Technical Background

For the sake of completeness and to establish our notation, we review the derivation of the equations we wish to solve. Consider the symmetric eigenvalue equation of structural dynamics:

$$[A]\{x_i\} = \lambda_i [B]\{x_i\} \tag{1}$$

in which $[A]$ and $[B]$ are $n \times n$ real, symmetric, and banded matrice, λ_i is a scalar eigenvalue, and $\{x_i\}$ the corresponding real eigenvector. If we adopt the eigenvector normalization condition

$$\lfloor x_i \rfloor [B]\{x_i\} = 1 \tag{2}$$

and differentiate Eqs. (1) and (2) with respect to a system scalar parameter r, we obtain

$$([A] - \lambda_i [B])\{x_i'\} = (\lambda_i'[B] + \lambda_i [B'] - [A'])\{x_i\} \equiv \{F_i\} \tag{3}$$

and

$$\lfloor x_i \rfloor [B]\{x_i'\} = -\tfrac{1}{2} \lfloor x_i \rfloor [B']\{x_i\} \equiv b \tag{4}$$

respectively, where $(\ )'$ denotes $\partial(\ )/\partial r$.

The eigenvalue derivative λ_i' on the right-hand side of Eq. (3) is obtained from the relationship

$$\lambda_i' = \lfloor x_i \rfloor ([A'] - \lambda_i [B'])\{x_i\} \tag{5}$$

which is obtained from the premultiplication of Eq. (3) by $\lfloor x_i \rfloor$ and substitution of Eq. (2), and the transpose of Eq. (1).

Assuming that $[A]$ and $[B]$ are known, that we have solved previously for λ_i, $\{x_i\}$, and λ_i', and that λ_i is not a repeated eigenvalue, Eqs. (3) and (4) comprise $n+1$ algebraic equations in n unknowns. Noting that $([A] - \lambda_i [B])$ is singular, it is realized that a redundant equation in the matrix system given by Eq. (3) must exist. The problem, however, is to determine which equation should be eliminated to remove the redundancy. Once this is established, it is a simple matter to remove this equation and solve the remaining system of n equations in a manner that takes as much advantage as possible of the bandwidth that existed in the $[A]$ and $[B]$ matrices prior to the removal of the redundant equation.

Technical Approach

Nelson[5] suggested that there are many candidate equations for removing the redundancy in $([A] - \lambda_i [B])$, but that the best would be one corresponding to the maximum element of eigenvector $\{x_i\}$. We shall call this the pth equation. His reasoning was based upon the fact that removing an equation whose corresponding eigenvector element was zero (or very small) might not involve an equation strongly coupled to the redundancy. Implicit in this idea was the unstated observation that the A matrix contained no zero on the diagonal (since it is positive definite).

Nelson then suggested that an efficient method to remove the pth equation is simply to zero out the pth row and the pth column of $([A] - \lambda_i [B])$, except for the diagonal element (which is set to an arbitrary constant), and to zero out the pth element of the right-hand side vector. This step produces an artificial zero for the pth element of the solution vector. This can be adjusted, as will be discussed later.

The resulting system of equations, which is equivalent to Eqs. (3) and (4) can then be written as

$$\left[\begin{array}{c|c} \backslash A_{11} \backslash & \{A_{12}\} \\ \hline A_{21} & A_{22} \end{array}\right] \left\{\begin{array}{c} \bar{x}' \\ \hline x_p' \end{array}\right\} = \left\{\begin{array}{c} \bar{F} \\ \hline b \end{array}\right\} \tag{6}$$

where:

1) $\{x_i'\}$ has been transformed into $\{\bar{x}'/x_p'\}$, for which the pth term of $\{x_i'\}$ (i.e., x_p') has been shifted to the $n+1$ position and all other terms are the same.

2) $\{F_i/b\}$ has been transformed into $\{\bar{F}/b\}$ where the pth term of $\{F_i\}$ has been made zero and all of the other terms remain the same.

3) $([A] - \lambda_i [B])$ has been transformed into the banded matrix $[\backslash A_{11} \backslash]$ in which the pth row and column are zero (except for the pth diagonal term) and all other terms remain the same.

4) $\{x_i\}^T [B]$ has been transformed into $\lfloor A_{21} \rfloor$.

5) $\{A_{12}\}$ was previously the pth column of $([A] - \lambda_i [B])$, except for the pth element now set to zero.

6) A_{22} was the previous pth element of the row matrix $\{x_i\}^T [B]$.

Three methods for solving Eq. (6) will now be discussed. We shall start with two new alternative procedures before proceeding to Nelson's method. The computational efficiency of the three methods is somewhat problem-dependent and will be discussed in the concluding section of this work.

Alternate Method A (Direct Procedure)

From Eq. (6),

$$\{\bar{x}'\} = [\backslash A_{11} \backslash]^{-1}(\{\bar{F}\} - x_p'\{A_{12}\}) \tag{7}$$

and

$$\lfloor A_{21} \rfloor \{\bar{x}'\} + A_{22} x_p' = b \tag{8}$$

Substituting Eq. (7) into Eq. (8) yields

$$\lfloor A_{21} \rfloor [\backslash A_{11} \backslash]^{-1}(\{\bar{F}\} - x_p'\{A_{12}\}) + A_{22} x_p' = b$$

or

$$(A_{22} - \lfloor A_{21} \rfloor [\backslash A_{11} \backslash]^{-1}\{A_{12}\}) x_p' = b - \lfloor A_{21} \rfloor [\backslash A_{11} \backslash]^{-1}\{\bar{F}\} \tag{9}$$

To solve Eq. (9) efficiently, we let

$$[\backslash A_{11} \backslash] = [L][L]^T \tag{10}$$

$$\lfloor A_{21} \rfloor [\backslash A_{11} \backslash]^{-1}\{A_{12}\} = \lfloor u \rfloor \{v\} \tag{11}$$

where $[L]$ is a lower triangular matrix.

We then use efficient forward and backward substitution of the system of equations

$$[L]\{v\} = \{A_{12}\} \tag{12}$$

$$\lfloor u \rfloor [L]^T = \lfloor A_{21} \rfloor \tag{13}$$

to solve for $\lfloor u \rfloor$ and $\{v\}$.

Having achieved this, we then substitute back into Eqs. (11) and (9) to obtain x_p'. Following this, we again use forward and backward substitution in conjunction with Eq. (7) to solve for $\{\bar{x}'\}$. (See Appendix A.)

The pth element of $\{\bar{x}'\}$, which will be zero, is then replaced by x_p', thus completing the solution..

Alternate Method B (Iterative Procedure)

From Eq. (6), it follows that

$$x'_p = \frac{1}{A_{22}}(b - \lfloor A_{21} \rfloor \{\bar{x}'\}) \tag{14}$$

$$[\backslash A_{11} \backslash]\{\bar{x}'\} = \{\bar{F}\} - x'_p\{A_{12}\} \tag{15}$$

Substituting Eq. (14) into Eq. (15) yields

$$[\backslash A_{11} \backslash]\{\bar{x}'\} = \{\bar{F}\} - \frac{1}{A_{22}}(b - \lfloor A_{21} \rfloor \{\bar{x}'\})\{A_{12}\}$$

or

$$[\backslash A_{11} \backslash]\{\bar{x}'\} = \{\bar{F}\} - \frac{b}{A_{22}}\{A_{12}\} + \frac{1}{A_{22}}\{A_{12}\} \lfloor A_{12} \rfloor \{\bar{x}'\} \tag{16}$$

The iteration index $\bar{r}$ is now introduced and Eq. (16) is replaced by the equation

$$[\backslash A_{11} \backslash]\{\bar{x}'\}_{\bar{r}} = \{\bar{F}\} - \frac{b}{A_{22}}\{A_{12}\} + \frac{1}{A_{22}}\{A_{12}\} \lfloor A_{21} \rfloor \{\bar{x}'\}_{\bar{r}-1} \tag{17}$$

Equation (17) is now solved iteratively for $\{\bar{x}'\}_{\bar{r}}$, using forward and backward solutions after decomposition of $[\backslash A_{11} \backslash]$ into $[L]\ [L]^T$ and starting with $\{\bar{x}'\}_0 = \{0\}$. (See Appendix B.)

Once again, as with alternate method A, the pth element of $\{\bar{x}'\}$ will be zero and must be replaced by x'_p from Eq. (14) to yield the desired solution.

Nelson's Method

Let

$$\{x'_i\} = \{V_i\} + c_i\{x_i\} \tag{18}$$

and substitute into Eq. (3) to obtain

$$([A] - \lambda_i[B])\{V_i\} + c_i([A] - \lambda_i[B])\{x_i\} = \{F_i\} \tag{19}$$

Making use of Eq. (1), Eq. (19) becomes

$$([A] - \lambda_i[B])\{V_i\} = \{F_i\} \tag{20}$$

To solve Eq. (20) we now transform it as follows:

$$[\backslash A_{11} \backslash]\{V_i\} = \{\bar{F}\} \tag{21}$$

which guarantees a nonsingular system, so that $\{V_i\}$ can be obtained and which makes the pth element of $\{V_i\}$ equal to zero.

Equation (18) is then substituted into Eq. (4) to yield

$$\lfloor x_i \rfloor\ [B](\{V_i\} + c_i\{x_i\}) = b$$

or, by making use of Eq. (2)

$$c_i = b - \lfloor x_i \rfloor\ [B]\{V_i\} \tag{22}$$

(See Appendix C.)

Repeated Roots

To keep matters simple, let λ_i be the repeated eigenvalue for only two orthonormal eigenvectors $\{x_i\}$ and $\{y_i\}$, i.e.,

$$[A]\{x_i\} = \lambda_i[B]\{x_i\} \tag{23}$$

$$[A]\{y_i\} = \lambda_i[B]\{y_i\} \tag{24}$$

$$\lfloor x_i \rfloor\ [B]\{x_i\} = 1 \tag{25}$$

$$\lfloor y_i \rfloor\ [B]\{y_i\} = 1 \tag{26}$$

$$\lfloor x_i \rfloor\ [B]\{y_i\} = 0 \tag{27}$$

It can be shown that the linear combination of eigenvectors $\{x_i\}$ and $\{y_i\}$ may be used to produce another solution, $\{z_i\}$, of Eqs. (23) and (25) i.e.,

$$\{z_i\} = \alpha\{x_i\} + (1-\alpha^2)^{1/2}\{y_i\} \tag{28}$$

for any value of α as long as $\alpha^2 \leq 1$, i.e.,

$$[A]\{z_i\} = \lambda_i[B]\{z_i\} \tag{29}$$

$$\lfloor z_i \rfloor\ [B]\{z_i\} = 1 \tag{30}$$

Derivatives for Repeated Roots

Differentiating Eq. (29) gives

$$([A] - \lambda_i[B])\{z'_i\} = \{F_i\} \tag{31}$$

where

$$\{F_i\} = -([A'] - \lambda_i[B'] - \lambda'_i[B])\{z_i\} \tag{32}$$

Premultiplication of Eq. (31) by the two-row matrix $[x_i y_i]^T$ gives the reduced eigenvalue problem

$$[x_i y_i]^T([A'] - \lambda_i[B'])[x_i y_i]\begin{Bmatrix} \alpha \\ (1-\alpha^2)^{1/2} \end{Bmatrix} = \lambda'_i \begin{Bmatrix} \alpha \\ (1-\alpha^2)^{1/2} \end{Bmatrix} \tag{33}$$

where we have made use of Eqs. (23-32).

Solution of Eq. (33) yields two values for λ'_i, which indicates that the repeated eigenvalue λ_i is capable of splitting along two distinct continuous paths as we continuously vary the parameters of $[A]$ and $[B]$. Chen and Pan[4] have shown that if λ_i is of multiplicity $L \geq 2$, the reduced eigenvalue equation for determining λ_i is generalized as follows:

$$[D]\{\zeta\} = \lambda'_i\{\zeta\} \tag{34}$$

where $[D]$ is an $L \times L$ matrix given by

$$[D] \equiv [W]^T([A'] - \lambda_i[B'])[W] \tag{35}$$

and $[W]$ is the modal matrix of orthonormalized eigenvectors corresponding to the repeated root λ_i.

Upon solution of Eq. (33) or (34) for λ'_i, the two eigenvectors and $\{F_i\}$ of Eqs. (31) and (32) may be determined. By virture of the repeated root λ_i, it is known that $([A] - \lambda_i[B])$ will be rank n-2 and the derivative of the normalization condition

$$\lfloor z_i \rfloor\ [B]\{z'_i\} = -\tfrac{1}{2}\lfloor z_i \rfloor\ [B']\{z_i\} \tag{36}$$

alone will be insufficient to determine $\{z'_i\}$.

Chen and Pan,[4] who used modal expansion techniques to evaluate eigenvector gradients, imply that the additional equations needed to solve for $\{z'_i\}$ are of the form

$$\lfloor z_{i+j} \rfloor\ [B]\{z'_i\} = 0, \quad j = 1,2,\ldots,L-1 \tag{37}$$

Although this equation is not derived in their work, it can be justified for cases in which

$$\lfloor z_{i+j} \rfloor [B'] \{z_i\} = 0, \quad j = 1,2,...,L-1 \tag{38}$$

This can be shown as follows: Consider the orthogonality condition

$$\lfloor z_{i+j} \rfloor [B] \{z_i\} = 0, \quad j = 1,2,...,L-1 \tag{39}$$

and obtain its derivative, i.e.,

$$\lfloor z'_{i+j} \rfloor [B] \{z_i\} + \lfloor z_{i+j} \rfloor [B'] \{z_i\} + \lfloor z_{i+j} \rfloor [B] \{z'_i\} = 0 \tag{40}$$

Equation (37) is a possible (but nonunique) condition of Eq. (40) for the special case when Eq. (38) is valid.

Thus, we see that the indeterminancy in obtaining unique eigenvector solutions for repeated roots carries over to obtaining the derivatives for such eigenvectors.

To solve for the gradients of eigenvectors corresponding to a repeated eigenvalue we shall, in the spirit of Nelson's work,[5] let

$$([A] - \lambda_i [B])[V_i V_{i+1}] = [F_i F_{i+1}] \tag{41}$$

where

$$\{z'_i\} = \{V_i\} + c_1 \{z_i\} + c_2 \{z_{i+1}\}$$

$$\{z'_{i+1}\} = \{V_{i+1}\} + d_1 \{z_i\} + d_2 \{z_{i+1}\} \tag{42}$$

After substitution of these equations into Eqs. (36) and (40), we obtain

$$c_1 = -\tfrac{1}{2} \{z_i\}^T [B'] \{z_i\} - \{z_i\}^T [B] \{V_i\} \tag{43}$$

$$d_2 = -\tfrac{1}{2} \{z_{i+1}\}^T [B'] \{z_{i+1}\} - \{z_{i+1}\}^T [B] \{V_{i+1}\} \tag{44}$$

and

$$c_2 + d_1 = -\{z_{i+1}\}^T [B] \{V_i\} - \{z_i\}^T [B] \{V_{i+1}\} - \{z_{i+1}\}^T [B'] \{z_i\} \tag{45}$$

Expanding Nelson's procedure for not destroying the bandedness of Eq. (31), we select two rows and columns (p and q) to eliminate, except in the diagonal elements, from $([A] - \lambda_i [B])$. Using reasoning similar to that used before, we select p to correspond to the maximum element of $\{z_i\}$ and q to correspond to the maximum element of $\{z_{i+1}\}$. If p and q happen to be equal, then we simply select q to correspond to the next largest element in $\{z_{i+1}\}$.

Having modified $([A] - \lambda_i [B])$ in this manner, the matrix will now be of rank n and nonsingular. Once we set both the pth and qth elements in $\{F_i\}$ and $\{F_{i+1}\}$ to zero this will yield zeroes for the pth and qth elements of both $\{V_i\}$ and $\{V_{i+1}\}$ once the modified Eq. (41) is solved.

Discussion

It should be noted that the vector $\{V_i\}$ [see Eq. (18)] is not devoid of additional components of the eigenvector $\{x_i\}$ and that c_i is not the total modal content of $\{x_i\}$ in $\{x'_i\}$. A second minor point with regard to Nelson's work is that the value of c_i given in his numerical example should read $-2/\sqrt{3}$, and not $-\tfrac{2}{3}$. These points are noted here because they did create some difficulty with regard to interpretation of this important work.

In solving for the eigenvector derivatives by any of the three methods presented in this paper, it is necessary to decompose the $n \times n$ banded matrix $[\diagdown A_{11} \diagdown]$ into $[L]\ [L]^T$. This will involve of the order of $n \cdot m^2$ operations, where m is the effective bandwidth of the matrix $([A] - \lambda_i [B])$. For Nelson's method, we must then perform additional operations of the order $n \cdot m$ to effect a solution. For alternate method A, we require approximately 2 $(n \cdot m)$ additional operations, or twice the additional operations required by Nelson's method. Alternate method B is iterative and, therefore, problem dependent.

It can be shown that the convergence rate of such iterative procedures depends upon the maximum eigenvalue, $\lambda_{G\max}$ of the matrix $[G]$, where

$$[G] = [\diagdown A_{11} \diagdown]^{-1} \{A_{12}\} \lfloor A_{21} \rfloor \tag{46}$$

Roughly speaking, this procedure will converge in s iterations, where s is related to the largest eigenvalue of $[G]$, and the convergence tolerance ϵ, where $\epsilon \sim (\lambda_{G\max})^s$.

Thus, if $\lambda_{G\max} = 0.5$ and $\epsilon = 0.01$ or 0.001, s will be 7 or 10, respectively. This procedure will require approximately four times the additional operations required by alternate method A. However, for the class of problems this paper addresses (i.e., large structural vibration problems for which $n \gg m$) it appears that the majority of the computations involve the decomposition of $[\diagdown A_{11} \diagdown]$ and a comparison of running times t for the methods discussed would yield

Nelson	$t \sim n \cdot m\ (m+1)$
Alternate A	$t \sim n \cdot m\ (m+2)$
Alternate B	$t \sim n \cdot m\ (m+8)$

It is seen that Nelson's method has a slight edge over alternate methods A and B, but that this advantage decreases significantly when $m \gg 1$. The alternate methods also involve more traditional concepts and may occasionally be easier to comprehend and work with.

With regard to the repeated eigenvalue problem, we see that, although the repeated eigenvalue derivatives are unique, it is not possible to uniquely determine the eigenvector gradients, just as it is not possible to uniquely determine the eigenvectors themselves. The implication here is that it is not sufficient to simply consider only one eigenvector or its derivative when they correspond to repeated eigenvalues. Rather, for a complete picture, it is necessary to consider all possible eigenvectors and the derivatives that correspond to a particular eigenvalue.

Extension of the repeated root gradient procedure presented herein, for eigenvalues of multiplicity greater than two, can be derived by letting

$$\{z'_i\} = \{V_i\} + c_1 \{z_i\} + c_2 \{z_{i+1}\} + c_3 \{z_{i+2}\} + \ldots$$

$$\{z'_{i+1}\} = \{V_{i+1}\} + d_1 \{z_1\} + d_2 \{z_{i+1}\} + d_3 \{z_{i+2}\} + \ldots \tag{47}$$

$$\{z'_{i+2}\} = \{V_{i+2}\} + e_1 \{z_i\} + e_2 \{z_{i+1}\} + e_3 \{z_{i+2}\} + \ldots$$

where

$$\{z_i\}, \ \{z_{i+1}\}, \ \{z_{i+2}\},\ldots$$

are the eigenvectors corresponding to the repeated eigenvalue λ_i and an additional equation must be removed from $([A] - \lambda_i [B])$ for each additional eigenvector.

It should be noted that the considerations and techniques discussed for accommodating repeated roots can also be employed if the eigenvalues of a given problem are so closely spaced as to create computational difficulties.

Appendix A

The computationally efficient steps to achieve alternate method A may be summarized as follows:

1) Decompose $[\backslash A_{11}\backslash]$ into $[L]\{L\}^T$.
2) Solve $[L]\{v\}=\{A_{12}\}$ for $\{v\}$.
3) Solve $[L]\{w\}=\{\bar{F}\}$ for $\{w\}$.
4) Solve $\lfloor u \rfloor \; [L]^T=\lfloor A_{21} \rfloor$ for $\lfloor u \rfloor$.
5) Compute x_p', where

$$x_p'=\frac{b-\lfloor u \rfloor \{w\}}{A_{22}-\lfloor u \rfloor \{v\}}$$

6) Solve $[L]^T\{\bar{x}'\}=\{w\}-x_p'\{v\}$ for $\{\bar{x}'\}$.

Appendix B

The computationally efficient steps to achieve alternate method B may be summarized as follows:

1) Decompose $[\backslash A_{11}\backslash]$ into $[L][L]^T$.
2) Solve $[L]\{v\}=\{A_{12}\}$ for $\{v\}$.
3) Solve $[L] \; \{w\}=\{\bar{F}\}$ for $\{w\}$.
4) Set $\{\bar{x}'\}_0=\{0\}$ and $s=0$ and solve

$$[L]^T\{\bar{x}'\}_{s+1}=\{w\}-\alpha_s\{A_{12}\}$$

for $\{x'\}_{s+1}$, where

$$\alpha_s=\frac{b-\lfloor A_{21} \rfloor \{\bar{x}'\}_s}{A_{22}}$$

5) After $\{\bar{x}'\}_{s+1}$ converges, set it equal to $\{\bar{x}'\}$ and compute x_p' where

$$x_p'=\frac{b-\lfloor A_{21} \rfloor \{\bar{x}'\}}{A_{22}}$$

Appendix C

The computationally efficient steps to achieve Nelson's method may be summarized as follows:

1) Decompose $[\backslash A_{11}\backslash]$ into $[L][L]^T$.
2) Solve $[L]\{w\}=\{\bar{F}\}$ for $\{w\}$.
3) Solve $[L]^T\{v_i\}=\{w\}$ for $\{v_i\}$.
4) Compute c_i where

$$c_i=b-\lfloor A_{21} \rfloor \{v\}$$

5) Form $\{x_i'\}$ from

$$\{x_i'\}=\{v_i\}+c_i\{x_i\}$$

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